Time-symmetric initial data for binary black holes in numerical relativity

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Abstract

We look for physically realistic initial data in numerical relativity which are in agreement with post-Newtonian approximations. We propose a particular solution of the time-symmetric constraint equation, appropriate to two momentarily static black holes, in the form of a conformal decomposition of the spatial metric. This solution is isometric to the post-Newtonian metric up to the 2PN order. It represents a non-linear deformation of the solution of Brill and Lindquist, i.e. an asymptotically flat region is connected to two asymptotically flat (in a certain weak sense) sheets, that are the images of the two singularities through appropriate inversion transformations. The total ADM mass M as well as the individual masses m_1 and m_2 (when they exist) are computed by surface integrals performed at infinity. Using second order perturbation theory on the Brill-Lindquist background, we prove that the binary's interacting mass-energy $M - m_1 - m_2$ is well-defined at the 2PN order and in agreement with the known post-Newtonian result.

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I. MOTIVATION AND RELATION TO OTHER WORKS

The numerical computation of the collision of two black holes is of paramount importance for the observation of gravitational waves by the network of laser-interferometric detectors. When investigating this problem the ten Einstein field equations are separated into: (i) four constraint equations that are to be satisfied by some initial data given on an initial three-dimensional Cauchy hypersurface; (ii) six hyperbolic-like equations describing the dynamical evolution of the gravitational field on neighbouring hypersurfaces. The Bianchi identities guarantee that the constraint equations are satisfied on neighbouring hypersurfaces if they are on the initial hypersurface. There are infinitely many ways that the initial data can be chosen to represent the starting state of the evolution of black holes. It is widely admitted that the problem of choosing physically realistic initial conditions for the collision of two black holes has not yet been solved. There has been a lot of concern in the litterature [1] for knowing what would really motivate physically a particular choice of initial data.

Let us consider the problem of *time-symmetric* initial data, which are physically appropriate to two momentarily static black holes, i.e. with zero initial velocities. The dynamical evolution of time-symmetric data describes the subsequent *head-on* collision of the two black holes. In this situation the constraint equations reduce to the Hamiltonian or scalar contraint equation $R = 0$ (in vacuum — the case appropriate to black holes), with R being the three-dimensional scalar curvature. Considering as usual [2, 3, 4] a conformal decomposition of the spatial metric (spatial indices $i, j, \cdots = 1, 2, 3$),

$$\gamma_{ij} = \Psi^4 \widetilde{\gamma}_{ij} \,, \tag{1.1}$$

where γ_{ij} is the physical metric and $\widetilde{\gamma}_{ij}$ denotes the conformal (unphysical) metric, we obtain the Lichnerowicz [2] equation, which is an elliptic-type equation to be satisfied by the conformal factor Ψ. In the time-symmetric case that equation becomes

$$\widetilde{\Delta}\Psi = \frac{\Psi}{8}\widetilde{R} \,. \tag{1.2}$$

The scalar curvature $\widetilde{R}$ and Laplacian $\widetilde{\Delta}$ are the ones associated with the conformal metric $\widetilde{\gamma}_{ij}$. The interest of the conformal decomposition is that by solving Eq. (1.2) we generate a physical solution γ_{ij} of the constraint equation starting from *any* choice for the conformal

metric $\widetilde{\gamma}_{ij}$ (*a priori*). Therefore the problem of initial conditions resides in the choice of a physically well-motivated conformal metric $\widetilde{\gamma}_{ij}$.

The simplest choice of initial conditions (one motivated by simplicity rather than by physics) is the one for which $\widetilde{\gamma}_{ij}$ equals the flat metric δ_{ij}. In that case Eq. (1.2) reduces to the flat-space Laplace equation. Some exact solutions, appropriate to (momentarily static) black holes, have been obtained by Misner [5] and Lindquist [6], and Brill and Lindquist [7]. The solution of Brill and Lindquist [for which the conformal factor Ψ^{BL} takes the form of Eq. (3.2b) below] is particularly interesting: it describes the "geometrostatics" of two black holes, consisting of three asymptotically flat regions connected by two Einstein-Rosen bridges (actually, the solution is known for N black holes). The one region containing the two throats is supposed to represent our universe, while the two sheets expanding behind it are associated with the two black holes. The beauty of the Brill-Lindquist solution is that not only the total ADM mass-energy M of the binary, but also the two individual masses m_1 and m_2 of the black holes, are computed "at infinity". One can use for instance standard surface integrals extending on topological two-spheres at infinity. The binary's geometrostatic energy (i.e. the gravitational interacting or binding energy, in the center-of-mass frame) is therefore computed unambiguously as $\frac{E}{c^2} = M - m_1 - m_2$.

Yet the solution of Brill and Lindquist, despite its undeniable interest, is not "physically realistic" in the sense that it differs, in the limit $c \to +\infty$, from the three-metric found for a post-Newtonian solution. Indeed the post-Newtonian metric generated by two point-particles is known to deviate from conformal flatness at the 2PN order (see e.g. Ref. [12]) [1]. In consequence both the Brill-Lindquist metric and the associated binding energy E disagree with the post-Newtonian results from the 2PN order.

A compelling motivation for constructing physically realistic initial data is the agreement with the post-Newtonian approximation when $c \to +\infty$. Recall that the post-Newtonian theory provides some explicit expressions for the metric, equations of motion and energy of binary systems of point-particles. The post-Newtonian metric is valid in the binary's "near-zone" (size of near-zone is much less than a gravitational wavelength), but has been proved to come from the re-expansion when $c \to +\infty$ of a "global" (post-Minkowskian-type) solution,

[1] In this paper c and G denote the speed of light and the gravitational constant. As usual the nPN order means the terms of order $\frac{1}{c^{2n}}$ when $c \to +\infty$.

defined everywhere in space-time including the wave zone [14]. Furthermore, in the post-Newtonian approach the modelling of black holes by point-like particles — i.e. technically by Dirac delta-functions in the stress-energy tensor — is rather well justified. We shall provide below some further evidence that the "post-Newtonian masses" of point-particles are indeed identical to the black-hole masses [2].

It has been suggested [16] (see also [17]) that, in order of taking into account the post-Newtonian physics into the initial data, one should adopt for the *conformal* metric $\widetilde{\gamma}_{ij}$ directly a post-Newtonian solution. In such a proposal one expects that by correcting the post-Newtonian solution by means of a conformal factor Ψ^4 (computed numerically), one will somehow be able to "compensate" in the physical metric γ_{ij} (i.e., in fact, to cancel out) the systematic higher-order post-Newtonian error terms that are neglected in $\widetilde{\gamma}_{ij}$.

There has been other proposals for realistic initial data, in particular built on the relaxation of the assumption of conformal flatness [1]. One of these is to use for $\widetilde{\gamma}_{ij}$ a linear combination of two (boosted versions of) Kerr-Schild metrics [18, 19]. It is not known if the physical metric which is generated in this way from the numerically-computed conformal factor, is consistent with post-Newtonian (e.g. 2PN) calculations.

On the other hand, we should remark that to which extent the hypothesis of conformal flatness introduces some unphysical spurious (and numerically important) effects remains an issue. Indeed, relaxing this hypothesis may not always be a panacea. A quite different idea for settling the initial conditions of two black holes is to solve numerically a subset of the Einstein field equations (the four constraint equations plus one evolution equation), under the two premices of conformal flatness and the existence of a helical Killing symmetry [20, 21]. Despite the conformal flatness of the spatial metric, the latter calculation gives results in good agreement with post-Newtonian predictions [21, 22, 23]. Nevertheless, we think that it is important, at some stage, to get rid of the hypothesis of conformal flatness.

We emphasize furthermore that the agreement between numerical relativity [20, 21] and post-Newtonian theory [21, 22, 23] holds up to the very relativistic regime of the innermost circular orbit (ICO), where the orbital velocities are of the order of 50% of the speed of light [3]. In Refs. [22] it is suggested that the result for the ICO of two black holes with

[2] See also, in a similar context, the matching of a 1PN solution for the orbital motion of point-particles to two perturbed Schwarzschild black holes [15].

[3] The ICO is defined by the minimum of the binary's energy function for circular orbits. It represents

comparable masses, at the 3PN approximation [4], is likely to be close to the "exact" solution, within 1% of fractional accuracy or better. This constitutes a motivation for advocating that the black-hole initial conditions, which are to be set (presumably) around the location of the ICO, should be in agreement with post-Newtonian theory.

In the present paper we propose an alternative way for incorporating the post-Newtonian information into the initial data of black-hole binaries (in the time-symmetric case). We find, in Section II, a simple expression for the conformal metric $\widetilde{\gamma}_{ij}$, which is such that the corresponding *physical* metric γ_{ij} is isometric (i.e. differs by a coordinate transformation) to the standard post-Newtonian spatial metric at the 2PN order in the limit $c \to +\infty$. At the same time, the solution is defined globally in space, with a global structure similar to the one of Brill and Lindquist. Our solution is not "exact", but exists as a certain non-linear perturbation, investigated in Section III, of the Brill-Lindquist solution, playing here the role of a "background" metric. Most importantly, in Section IV we investigate the asymptotic structure of the solution, and compute "geometrically" the masses M and m_1, m_2, i.e. by surface integrals performed in their respective domains at infinity. In Section V the binary's interacting energy, deduced from the previous masses, is proved to be in agreement with the known post-Newtonian energy up to the 2PN order. (We shall find, however, that the definition we adopt for the two individual masses m_1 and m_2 makes sense only up to the 2PN order.)

II. CONFORMAL DECOMPOSITION OF THE SPATIAL METRIC

A. Definition of the conformal metric

The conformal metric we propose, appropriate to two black holes of masses m_1 and m_2 (to agree later with the "geometrical" masses), located at the singular points $\mathbf{y}_1$ and $\mathbf{y}_2$, and

a useful reference point for the numerical [20, 21] and 3PN [22] calculations because for both of them the ICO is well-defined and can be meaningfully compared. However, the radiation reaction terms are neglected in its definition, so the ICO probably does not have a rigorous physical meaning in a context of exact radiative solutions.

[4] We mean the standard Taylor-expanded form of the approximation — without using any post-Newtonian resummation techniques.

momentarily at rest ($\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}$), is

$$\widetilde{\gamma}_{ij} = \delta_{ij} - \frac{8G^2 m_1 m_2}{c^4} \frac{\partial^2 g}{\partial y_1^{<i} \partial y_2^{j>}} \ . \tag{2.1}$$

The function g introduced here represents an elementary "kernel" playing an important role in post-Newtonian calculations [8, 9, 10, 11, 12, 13]. It depends on the "field" point $\mathbf{x}$ on the one hand, and on the pair of "source" points $\mathbf{y}_1$, $\mathbf{y}_2$ on the other hand; it is defined by

$$g(\mathbf{x}; \mathbf{y}_1, \mathbf{y}_2) = \ln \left(r_1 + r_2 + r_{12} \right) \ , \tag{2.2}$$

where $r_1 = |\mathbf{x} - \mathbf{y}_1|$ and $r_2 = |\mathbf{x} - \mathbf{y}_2|$ are the distances to the black holes, and $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$ is their separation. The function g satisfies, in the sense of distributions (i.e. for all values of $\mathbf{x}$, including the singular points $\mathbf{y}_1$ and $\mathbf{y}_2$),

$$\Delta g = \frac{1}{r_1 r_2} \ , \tag{2.3}$$

where Δ denotes the usual flat-space Laplacian with respect to the field point $\mathbf{x}$. In Eq. (2.1) the derivatives are taken with respect to the two source points [5]. The carets around the indices refer to the symmetric and trace-free (STF) projection: $T_{<ij>} \equiv \frac{1}{2}\left(T_{ij} + T_{ji}\right) - \frac{1}{3}\delta_{ij} T_{kk}$; so the trace of the metric (2.1) is normalized to be $\widetilde{\gamma}_{ii} = 3$. It is of course quite natural (and useful in practice) to impose that the deviation of the conformal metric from flat space be trace-free.

Our proposal is to generate, by means of numerical techniques (i.e. with the help of elliptic solvers), a conformal factor Ψ solving the constraint equation (1.2) corresponding to the particular choice of conformal metric (2.1). The metric $\gamma_{ij} = \Psi^4 \widetilde{\gamma}_{ij}$ we obtain in this way will incorporate the post-Newtonian (2PN) physics of the initial state of the head-on collision of two black holes [6]. The latter assertion will now be proved, for the rest of the paper, with the help of analytic perturbation methods.

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[5] The following explicit formula holds:

$${}_i g_j \equiv \frac{\partial^2 g}{\partial y_1^i \partial y_2^j} = \frac{n_{12}^i n_{12}^j - \delta^{ij}}{r_{12}(r_1 + r_2 + r_{12})} + \frac{(n_{12}^i - n_1^i)(n_{12}^j + n_2^j)}{(r_1 + r_2 + r_{12})^2} \ .$$

Here, $n_1^i = (x^i - y_1^i)/r_1$, $n_2^i = (x^i - y_2^i)/r_2$ and $n_{12}^i = (y_1^i - y_2^i)/r_{12}$. See Ref. [12] for further discussion and formulas concerning the function g.

[6] See Refs. [24, 25] for numerical calculations of the head-on collision of black holes; see also Ref. [26] for a post-Newtonian calculation.

B. Relation with the post-Newtonian metric

The first result shows that the "near-zone" behaviour of the solution (i.e. $r/c \to 0$) is physically sound.

Theorem 1 *The post-Newtonian expansion (when $c \to +\infty$) of the solution $\gamma_{ij} = \Psi^4 \widetilde{\gamma}_{ij}$ of the constraint equation (1.2), where $\widetilde{\gamma}_{ij}$ is defined by Eq. (2.1), differs from the standard post-Newtonian spatial metric, calculated by standard post-Newtonian methods, by a mere change of coordinates at the* 2PN *order, i.e.*

$$\gamma_{ij} = \mathrm{g}_{ij}^{\mathrm{2PN}}\Big|_{\mathbf{v}_1=\mathbf{v}_2=\mathbf{0}} + \partial_i \xi_j + \partial_j \xi_i + \mathcal{O}\left(\frac{1}{c^6}\right) . \tag{2.4}$$

To be precise, by $\mathrm{g}_{ij}^{\mathrm{2PN}}$ we mean the spatial metric in *harmonic coordinates*, when truncated at the 2PN order, that is given by Eq. (7.2c) in Ref. [12]. As indicated in Eq. (2.4), we must set the particles' velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ to zero in the post-Newtonian metric in order to conform with the assumption of time-symmetry. The remainder $\mathcal{O}\left(\frac{1}{c^6}\right)$ stands for the neglected 3PN and higher-order terms.

The proof of Theorem 1 is easily achieved on the basis of a post-Newtonian iteration of Eq. (1.2). At the 2PN order, this equation becomes

$$\Delta\Psi = -\frac{G^2 m_1 m_2}{c^4} \partial_{ij} \left({}_{<i}g_{j>} \right) + \mathcal{O}\left(\frac{1}{c^6}\right) , \tag{2.5}$$

where we denote ${}_i g_j \equiv \frac{\partial^2 g}{\partial y_1^i \partial y_2^j}$. The most general solution reads as

$$\Psi = \psi - \frac{G^2 m_1 m_2}{2c^4} \mathrm{D} \left(\frac{g}{3} + \frac{r_1 + r_2}{2 r_{12}} \right) + \mathcal{O}\left(\frac{1}{c^6}\right) , \tag{2.6}$$

where ψ represents a solution of the homogeneous equation (i.e. $\Delta\psi = 0$)[7]. By comparing with the post-Newtonian metric (see Eq. (7.2c) in Ref. [12] in which $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}$), we

[7] We employ the notational shorthand $\mathrm{D} \equiv \frac{\partial^2}{\partial y_1^i \partial y_2^i}$ (so that $\mathrm{D}g = {}_i g_i$). Notice the useful relations

$$\begin{aligned} \mathrm{D}g &= \frac{1}{2r_1 r_2} - \frac{1}{2 r_1 r_{12}} - \frac{1}{2 r_2 r_{12}} , \\ \partial_{ij} \left({}_i g_j \right) &= \mathrm{D} \left[\frac{1}{2r_1 r_2} + \frac{1}{2 r_1 r_{12}} + \frac{1}{2 r_2 r_{12}} \right] . \end{aligned}$$

readily find that the latter solution is uniquely specified, at the 2PN order, as being

$$\psi = 1 + \frac{Gm_1}{2c^2 r_1}\left(1 - \frac{Gm_2}{2c^2 r_{12}}\right) + \frac{Gm_2}{2c^2 r_2}\left(1 - \frac{Gm_1}{2c^2 r_{12}}\right) + \mathcal{O}\left(\frac{1}{c^6}\right) . \tag{2.7}$$

Also uniquely determined is the expression of the vector ξ_i in Eq. (2.4), which represents an infinitesimal gauge transformation. At the 2PN order it takes the expression

$$\xi_i = \frac{G^2 m_1^2}{4c^4}\frac{n_1^i}{r_1} + \frac{G^2 m_2^2}{4c^4}\frac{n_2^i}{r_2} + \mathcal{O}\left(\frac{1}{c^6}\right) , \tag{2.8}$$

where $\mathbf{n}_1 \equiv (\mathbf{x} - \mathbf{y}_1)/r_1$ and $\mathbf{n}_2 \equiv (\mathbf{x} - \mathbf{y}_2)/r_2$. The system of spatial coordinates employed in the present paper, i.e. which corresponds to a conformal metric of the particular form displayed by Eq. (2.1), is thereby determined, and related up to the 2PN order to the harmonic coordinate system used in Ref. [12], by the coordinate change $\delta x^i = \xi^i$ (with $\xi^i = \delta^{ij}\xi_j$).

Summarizing this section, we have found a conformal decomposition of the metric γ_{ij} that is in agreement with the post-Newtonian metric up to the 2PN order. Notice, however, that the conformal metric we propose is not unique, because we can always add higher-order terms (e.g. 3PN) to Eq. (2.1). *A contrario*, this means that with the metric (2.1) one should not expect Theorem 1 to hold at the 3PN order and beyond. At the 3PN order for instance the conformal metric will be more complicated than the simple expression (2.1).

III. PERTURBATION OF THE BRILL-LINDQUIST SOLUTION

Having checked the near-zone structure of the metric, let us investigate some of its *global* properties, when it is viewed as a solution of the constraint equation (1.2) that is *a priori* valid everywhere on a space-like hypersurface. For this purpose, we impose that γ_{ij} is a non-linear perturbation of the exact solution of Brill and Lindquist [7] — which is conformally flat ($\widetilde{\gamma}_{ij}^{\mathrm{BL}} = \delta_{ij}$). This will mean that the topology of our solution is identical to the topology of the Brill-Lindquist solution, i.e. be "three-sheeted", in contrast with the two-sheeted topology of the Misner-Lindquist solution [5, 6].

A. Hierarchy of perturbation equations

Let us write Eq. (2.1) in a more transparent form, $\widetilde{\gamma}_{ij} = \delta_{ij} + \beta s_{ij}$, where

$$\beta \equiv -\frac{8G^2 m_1 m_2}{c^4}, \tag{3.1a}$$
$$s_{ij} \equiv \frac{\partial^2 g}{\partial y_1^{<i} \partial y_2^{j>}} \quad \text{(such that } s_{ii} = 0) . \tag{3.1b}$$

It is helpful to view the non-linear perturbation we want to consider, as being generated by the "seed" or "generating" function s_{ij}, and to interpret the parameter β as the magnitude of that perturbation. In the following, β will play the role of a "book-keeping" parameter allowing us to label the successive non-linear perturbation orders. Besides the non-conformally flat piece of the metric brought about by s_{ij}, it is evident that we must also introduce a perturbation in the conformal factor. We pose

$$\Psi = \psi + \sigma , \tag{3.2a}$$
$$\psi = 1 + \frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2} , \tag{3.2b}$$

where ψ denotes the Brill-Lindquist conformal factor [7], parametrized by two constants α_1 and α_2 (in the case of two particles). The quantity σ denotes a certain perturbation of ψ.

With full generality — within the present perturbative framework —, we look for the expression of σ in the form of an infinite power series in β, solving the equation (1.2):

$$\sigma = \sum_{n=1}^{+\infty} \beta^n \sigma_{(n)} . \tag{3.3}$$

We insist that in our approach the Brill-Lindquist solution plays the role of the *background* metric (it depends solely on α_1 and α_2), while the function s_{ij} yields a non-linear deformation of this background, perturbatively ordered by the book-keeping parameter β. For a given choice of s_{ij}, we expect that the resulting solution is unique (at least in a sense of formal power series in β). Because β involves two mass factors m_1 and m_2, so it is proportional to G^2, the perturbation series we look for is like a "double" post-Minkowskian expansion, going "twice as fast" as the usual post-Minkowskian expansion when $G \to 0$. Considering this series on the point of view of a post-Newtonian re-expansion, we see that each non-linear order brings in a new factor $1/c^4$, so that our perturbation series can be said to go by — quite efficient indeed — steps of 2PN orders.

Let us compare our definitions (3.2)-(3.3) with the result (2.6)-(2.7) provided by the agreement with the 2PN metric. We find that the constants α_1, α_2 are determined with relative 1PN accuracy,

$$\alpha_1 = \frac{Gm_1}{2c^2}\left[1 - \frac{Gm_2}{2r_{12}c^2} + \mathcal{O}\left(\frac{1}{c^4}\right)\right] \quad \text{and} \quad 1 \leftrightarrow 2 \ , \tag{3.4}$$

and that with this accuracy they agree with the prediction of the Brill-Lindquist solution [8]. In addition, we find that the perturbation σ in the conformal factor is given by the second term in the right side of Eq. (2.6), which comes in only at the 2PN order — it is purely linear in β. Hence,

$$\sigma_{(1)} = \frac{1}{16}\mathrm{D}\left(\frac{g}{3} + \frac{r_1 + r_2}{2r_{12}}\right) + \mathcal{O}\left(\frac{1}{c^2}\right) \ , \tag{3.5}$$

with all higher-order $\sigma_{(n)}$'s being negligible with this approximation. Actually, Eqs. (3.4)-(3.5) correspond to a particular "sharing" of the terms between α_1, α_2 on the one hand, and $\sigma_{(1)}$ on the other hand, since we can always add to $\sigma_{(1)}$ some "homogeneous" terms $\sim 1/r_1$ and $\sim 1/r_2$ without changing the equation for $\sigma_{(1)}$. It is obvious that such a sharing is physically irrelevant. However we shall forbid a different sharing of terms by adopting below the prescription (which represents simply a convenient choice) that $\sigma_{(1)}$ and all subsequent iterations solve the equation for the conformal factor in the sense of distributions. Some results more complete than (3.4)-(3.5) will be obtained below.

By inserting the perturbation ansatz (3.3) into the constraint equation (1.2), and by identifying each of the coefficients of the successive powers of β in both sides of the equation, we obtain a hierarchy (indexed by $n \in \mathbb{N}$) of Poisson-type equations:

$$\Delta\sigma_{(n)} = \Sigma_{(n)}\left[\sigma_{(1)}, \cdots, \sigma_{(n-1)}\right] \ , \tag{3.6}$$

where we recall that $\Delta \equiv \partial_i\partial_i$. The nth-order source term $\Sigma_{(n)}$ depends on the solutions

[8] We recall that α_1^{BL} and α_2^{BL} in the case of the Brill-Lindquist solution are related to the masses by the exact relations

$$\begin{aligned} \alpha_1^{\mathrm{BL}}\left[1 + \frac{\alpha_2^{\mathrm{BL}}}{r_{12}}\right] &= \frac{Gm_1}{2c^2} \quad \text{and} \quad 1 \leftrightarrow 2 \ , \\ \alpha_1^{\mathrm{BL}} + \alpha_2^{\mathrm{BL}} &= \frac{GM}{2c^2} \ . \end{aligned}$$

of the preceding iterations $\sigma_{(1)}, \cdots, \sigma_{(n-1)}$, on the "generating" function s_{ij} and on the "background" conformal factor ψ. Notice that ψ satisfies the equation $\Delta\psi = -4\pi(\alpha_1\delta_1 + \alpha_2\delta_2)$, where e.g. $\delta_1 \equiv \delta(\mathbf{x}-\mathbf{y}_1)$ denotes the Dirac function at the point $\mathbf{y}_1$. Obviously we may include the equation for the background conformal factor into our hierarchy of equations by posing $\Psi = \sum_{n=0}^{+\infty} \beta^n \sigma_{(n)}$, with $\sigma_{(0)} = \psi$ and $\Sigma_{(0)} = -4\pi(\alpha_1\delta_1 + \alpha_2\delta_2)$.

B. Analytic closed form of the linearized solution

To the linearized order the perturbation equation reads explicitly

$$\Delta\sigma_{(1)} = \frac{1}{8}\psi\partial_{ij}s_{ij} + \partial_j s_{ij}\partial_i\psi + s_{ij}\partial_{ij}\psi \ . \tag{3.7}$$

Interestingly, this equation turns out to be solvable in analytic closed form. We find that the *unique* solution of Eq. (3.7), that is valid in the sense of distributions and tends to zero at spatial infinity (i.e. when $r \equiv |\mathbf{x}| \to +\infty$), is

$$\begin{aligned}
\sigma_{(1)} \ =& \ \frac{1}{16}\mathrm{D}\left(\frac{g}{3} + \frac{r_1+r_2}{2r_{12}}\right) \\
+& \ \alpha_1\left\{\frac{H_1}{2} + \frac{K_1}{32} - \frac{1}{32}\mathrm{D}\left(\frac{1}{r_2}\ln\left[\frac{r_1}{r_{12}}\right]\right) + \frac{9}{32}\mathrm{D}\left(\frac{\ln r_1}{r_{12}}\right) + \frac{\mathrm{D}g}{12r_1}\right. \\
&\left. + \frac{1}{4}\Delta_1\left(\frac{g}{r_{12}}\right) - \frac{1}{32}\Delta_2\left(\frac{g}{r_{12}}\right) - \frac{1}{24r_1r_{12}^2} + \frac{1}{32r_2r_{12}^2}\right\} \\
+& \ \alpha_2\Big\{1 \leftrightarrow 2\Big\} \ .
\end{aligned} \tag{3.8}$$

Besides the already met shorthand $\mathrm{D} \equiv \frac{\partial^2}{\partial y_1^i \partial y_2^i}$, we denote the Laplacians with respect to the source points by $\Delta_1 \equiv \frac{\partial^2}{\partial y_1^i \partial y_1^i}$ and $\Delta_2 \equiv \frac{\partial^2}{\partial y_2^i \partial y_2^i}$. In Eq. (3.8), the two terms proportional to α_1 and α_2 are deduced from each other by label exchange $1 \leftrightarrow 2$. The solution involves the special functions H_1 and K_1 (and $1 \leftrightarrow 2$) which were introduced in Refs. [10, 12] for solving some elementary equations (in the sense of distributions) in the problems of equations of motion and wave generation at the 2PN order. We have,

$$\Delta H_1 \ = \ 2\,{}_ig_j\,\partial_{ij}\left(\frac{1}{r_1}\right) \ , \tag{3.9a}$$

$$\Delta K_1 \ = \ 2\,\mathrm{D}^2\left(\frac{\ln r_1}{r_2}\right) \ , \tag{3.9b}$$

in which ${}_i g_j \equiv \frac{\partial^2 g}{\partial y_1^i \partial y_2^j}$. The functions H_1 and K_1 admit the closed-form expressions given by Eqs. (3.48)-(3.51) in Ref. [10], or equivalently by Eqs. (6.3)-(6.5) in Ref. [12]; for the present purpose we adopt the formulas [9]

$$\begin{aligned}
H_1 &= \Delta_1 \left[\frac{g}{2r_{12}} + \mathrm{D}\Big(\frac{r_1+r_{12}}{2} g\Big)\right] - \mathrm{D}\left(\frac{\ln r_{12}}{r_1}\right) - \frac{3}{2}\mathrm{D}\left(\frac{\ln r_1}{r_{12}}\right) \\
&- \frac{r_2}{2r_1^2 r_{12}^2} + \frac{1}{2r_1^2 r_{12}} - \frac{1}{2r_1 r_{12}^2} , \qquad (3.10\text{a}) \\
K_1 &= \mathrm{D}\left(\frac{1}{r_2}\ln\left[\frac{r_1}{r_{12}}\right]\right) - \frac{1}{2r_1^2 r_2} + \frac{1}{2r_2 r_{12}^2} + \frac{r_2}{2r_1^2 r_{12}^2} . \qquad (3.10\text{b})
\end{aligned}$$

Anyway, we find that the exact solution of the linearized perturbation equation is

$$\begin{aligned}
\sigma_{(1)} &= \frac{1}{16}\mathrm{D}\left(\frac{g}{3} + \frac{r_1+r_2}{2r_{12}}\right) \\
&+ \alpha_1 \left\{\frac{1}{2}\Delta_1\left[\frac{g}{r_{12}} + \mathrm{D}\left(\frac{r_1+r_{12}}{2} g\right)\right] - \frac{1}{2}\mathrm{D}\left(\frac{\ln r_{12}}{r_1}\right) - \frac{15}{32}\mathrm{D}\left(\frac{\ln r_1}{r_{12}}\right) + \frac{\mathrm{D}g}{12 r_1}\right. \\
&\quad \left. -\frac{1}{32}\Delta_2\left(\frac{g}{r_{12}}\right) - \frac{15 r_2}{64 r_1^2 r_{12}^2} + \frac{1}{4r_1^2 r_{12}} - \frac{1}{64 r_1^2 r_2} - \frac{7}{24 r_1 r_{12}^2} + \frac{3}{64 r_2 r_{12}^2}\right\} \\
&+ \alpha_2\Big\{1 \leftrightarrow 2\Big\} . \qquad (3.11)
\end{aligned}$$

Let us quote also, for completeness, the fully explicit form obtained by expanding all the derivatives in this result:

$$\begin{aligned}
\sigma_{(1)} &= \frac{1}{96 r_1 r_2} + \frac{r_1+r_2}{64 r_{12}^3} + \frac{1}{192 r_{12}}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) - \frac{1}{64 r_{12}^3}\left(\frac{r_1^2}{r_2} + \frac{r_2^2}{r_1}\right) \\
&+ \alpha_1 \left\{-\frac{1}{4r_1^3} - \frac{13}{64 r_{12}^3} - \frac{1}{24 r_1 r_{12}^2} - \frac{5}{192 r_1^2 r_{12}} + \frac{5}{192 r_1^2 r_2} - \frac{r_1}{32 r_2 r_{12}^3} + \frac{3}{64 r_2 r_{12}^2}\right. \\
&\quad \left. -\frac{1}{24 r_1 r_2 r_{12}} + \frac{r_2}{4 r_1 r_{12}^3} - \frac{15 r_2}{64 r_1^2 r_{12}^2} + \frac{r_2}{4 r_1^3 r_{12}} + \frac{15 r_2^2}{64 r_1^2 r_{12}^3} + \frac{r_2^2}{4 r_1^3 r_{12}^2} - \frac{r_2^3}{4 r_1^3 r_{12}^3}\right\}
\end{aligned}$$

[9] We notice here (since it was not noticed in Refs. [10, 12]) that K_1 can also be given an interesting form in connection with a simple elementary kernel k_1, *viz*

$$\begin{aligned}
\Delta k_1 &= \frac{1}{r_1^2 r_2} , \\
K_1 &= \mathrm{D}\left(\frac{1}{r_2}\ln\left[\frac{r_1}{r_{12}}\right] - k_1\right) .
\end{aligned}$$

$$+\ \alpha_2\Big\{1 \leftrightarrow 2\Big\}\ . \tag{3.12}$$

The terms in the first line of Eqs. (3.11) or (3.12) contribute at the 2PN order in the conformal factor and they are in agreement with Eq. (3.5). The other terms, proportional to α_1 or α_2, will not contribute before the 3PN order [because α_1 and α_2 carry a factor $1/c^2$ in front]. Therefore, only a small part of the linearized approximation, the one given by the first term in (3.11), is necessary in the proof of Theorem 1. As a matter of fact, the non-linear perturbation we consider contains much more information than a mere post-Newtonian (2PN) expansion. Theorem 1 does not constitute a very stringent requirement on the non-linear solution of the perturbation equations.

A point we make by writing the linear solution $\sigma_{(1)}$ into the primary form (3.8), involving the intermediate functions H_1 and K_1, is that the latter functions do enter in the post-Newtonian metric at the 3PN order — with the same numerical coefficients as predicted by Eq. (3.8). This can be inferred from the expression of the 3PN spatial metric given by Eq. (111) in Ref. [14], which contains the particular non-linear potential called $\hat{X}$, together with the way that the potential $\hat{X}$ contains the functions H_1 and K_1 as calculated by Eq. (6.11) in [12]. Thus, although our solution agrees with post-Newtonian calculations up to the 2PN order only, it does contain some correct 3PN features.

C. Quadratic and higher-order approximations

At the level of the second-order perturbation ($\propto \beta^2$) the equation reads

$$\begin{aligned}\Delta\sigma_{(2)} \;=\;& \frac{1}{8}\sigma_{(1)}\partial_{ij}s_{ij} + \partial_j s_{ij}\partial_i\sigma_{(1)} + s_{ij}\partial_{ij}\sigma_{(1)} \\ &+ \frac{1}{8}\bigg(s_{ij}\Delta s_{ij} - 2s_{ij}\partial_i\partial_k s_{jk} - \partial_j s_{ij}\partial_k s_{ik} + \frac{3}{4}\partial_k s_{ij}\partial_k s_{ij} - \frac{1}{2}\partial_k s_{ij}\partial_i s_{jk}\bigg)\psi \\ &+ \bigg(-s_{ij}\partial_k s_{jk} - s_{jk}\partial_k s_{ij} + \frac{1}{2}s_{jk}\partial_i s_{jk}\bigg)\partial_i\psi - s_{ik}s_{jk}\partial_{ij}\psi\ .\end{aligned} \tag{3.13}$$

At the next level, cubic-order, the equation will be made of the same terms as in Eq. (3.13) but with the replacements $\sigma_{(1)} \to \sigma_{(2)}$ and $\psi \to \sigma_{(1)}$, together with many other terms that are purely cubic in s_{ij}. And so on for the higher-order equations.

We shall see in Section IV that in order to prove the agreement with the 2PN binary's *energy* (in contrast with the 2PN *metric*), we need the full information content about the

linearized solution given by Eqs. (3.11)-(3.12), and also a crucial piece coming from the *second-order* perturbation, solution of Eq. (3.13). In the case of non-linear perturbations ($n \geq 2$), it is in general impossible to find a solution in analytic closed form. Fortunately, what we shall need is only to control the expansion of $\sigma_{(2)}$ at spatial infinity (i.e. in the far zone, $r \to +\infty$). And this *can* be achieved, without disposing of the exact expression of $\sigma_{(2)}$, from the knowledge of the far-zone expansion of the corresponding source-term $\Sigma_{(2)}$. More generally, the far-zone expansion of the solution $\sigma_{(n)}$ for any n can be obtained on condition that the far-zone expansion of its source $\Sigma_{(n)}$ has been determined beforehand (say, by induction on n).

The method is issued from the investigation, in Refs. [27, 28], of the multipole expansion of the solution of a Poisson equation with non-compact-support source (i.e. whose support is $\mathbb{R}^3$). In the present context, the multipole expansion is completely equivalent to the far-zone expansion, when $r \to +\infty$. Following Eq. (C.9) in Ref. [28], we obtain the (formal) multipole expansion of $\sigma_{(n)}$ in the form [10]

$$\mathcal{M}(\sigma_{(n)}) \;=\; FP\left\{\Delta^{-1}\left[\mathcal{M}(\Sigma_{(n)})\right] - \frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}\right)\int_{\mathbb{R}^3} d^3\mathbf{y}\, y_L \Sigma_{(n)}(\mathbf{y})\right\}\,, \quad (3.14)$$

where the calligraphic letter $\mathcal{M}$ refers to the multipole or equivalently the far-zone expansion. The first term in the right side of (3.14) represents the effect of integrating "term by term" the multipole expansion of the source $\Sigma_{(n)}$ (this term exists only in the case of non-compact-support sources, because the multipole expansion of a compact-support function is zero). The second term in Eq. (3.14) is parametrized by the "multipole moments" associated with the solution, which are given by some definite integrals extending over the non-compact support $\mathbb{R}^3$ of $\Sigma_{(n)}$. The symbol *FP* acts on both terms of (3.14), and stands for a certain operation of taking the *Finite Part*, defined in Refs. [27, 28] by means of a process of complex analytic continuation (with parameter $B \in \mathbb{C}$). The finite part has proved to play a crucial role, in the case of non-compact-support sources like $\Sigma_{(n)}$, in order to ensure the well-definiteness of the integrals giving the multipole moments in (3.14). Notice that Eq. (3.14) has been proved, in Refs. [27, 28], to hold in the case of a regular source [i.e. $C^\infty(\mathbb{R}^3)$].

[10] Technical notations in Eq. (3.14) are Δ^{-1} for the standard Poisson integral; $L \equiv i_1 \cdots i_\ell$ for a multi-index composed of ℓ indices; $\partial_L \equiv \partial_{i_1}\cdots\partial_{i_\ell}$ for the product of ℓ partial derivatives; $y_L \equiv y_{i_1}\cdots y_{i_\ell}$ for the product of ℓ spatial vectors. We do not write the ℓ summation symbols over the ℓ indices composing L.

Nevertheless, in the presence of the singular points $\mathbf{y}_1$ and $\mathbf{y}_2$, the formula (3.14) can still be applied, but only in the case of that solution $\sigma_{(n)}$ for which $\Delta\sigma_{(n)} = \Sigma_{(n)}$ is satisfied in the sense of distributions.

IV. TOTAL MASS AND INDIVIDUAL BLACK-HOLE MASSES

A. Asymptotic structure of the solution

A central result of the present paper concerns the "asymptotics" of our solution — both at spatial infinity and in the vicinity of the two particles.

Theorem 2 *(i) The metric γ_{ij} is asymptotically flat at infinity (when $r \equiv |\mathbf{x}| \to +\infty$), i.e.*

$$\gamma_{ij} = \delta_{ij} + \mathcal{O}\left(\frac{1}{r}\right) . \tag{4.1}$$

(ii) The two singular points $\mathbf{y}_1$ and $\mathbf{y}_2$ are the images, via some appropriate inversions of the radial coordinates: $\rho_1 \sim 1/r_1$ and $\rho_2 \sim 1/r_2$, of two "asymptotically finite" regions (when $\rho_1 \to +\infty$ and $\rho_2 \to +\infty$), in the sense that the metric, in coordinates $(\rho_1, \mathbf{n}_1)$, behaves like

$$\Gamma_{ij}(\rho_1, \mathbf{n}_1) = \delta_{ij} + \pi_{ij}(\mathbf{n}_1) + \mathcal{O}\left(\frac{1}{\rho_1}\right) , \tag{4.2}$$

where π_{ij} denotes a certain function of the angles.

Theorem 2 says that the solution is composed of an asymptotically flat universe connected by continuity, *via* some Einstein-Rosen-like bridges, to two other regions which are asymptotically finite in the sense of Eq. (4.2). Note that, though the metric (4.2), unlike the corresponding Brill-Lindquist metric, is not asymptotically flat in the vicinity of the two particles (*stricto-sensu*), the violation of asymptotic flatness is not very severe, because the metric tends toward a constant with respect to ρ_1, and does not involve any divergency "at infinity": it remains asymptotically finite — hence the name. On the other hand, what is very important is that the *real* universe, as depicted by this solution, *is* asymptotically flat at spatial infinity in the usual sense of Eq. (4.1). The global structure described by Theorem 2 represents an attractive feature, we argue, for considering the solution as a physically well-motivated initial state of binary black holes.

The asymptotic flatness when $r \to +\infty$ follows from the far-zone expansion of the generating function (3.1b), which is easily checked to start at $s_{ij} = \mathcal{O}\left(\frac{1}{r}\right)$. Using this fact we find that the source term for the linearized perturbation — i.e. the right side of Eq. (3.7) — behaves like $\Sigma_{(1)} = \mathcal{O}\left(\frac{1}{r^3}\right)$. With the help of Eq. (3.14) we readily obtain $\sigma_{(1)} = \mathcal{O}\left(\frac{1}{r}\right)$, as can also be checked directly with our exact results (3.11)-(3.12). Then, similarly, we deduce, by induction on the non-linear order n [using Eq. (3.14) at each step], that $\sigma_{(n)} = \mathcal{O}\left(\frac{1}{r}\right)$ for any n. So $\sigma = \mathcal{O}\left(\frac{1}{r}\right)$, and the result follows.

The main point about Theorem 2 is the behaviour of the solution in the vicinity of the two particles. When $r_1 \to 0$, we find that s_{ij} admits a *bounded* expansion [i.e. s_{ij} does not blow up when $r_1 \to 0$], of the type

$$\beta s_{ij} = \varepsilon_{ij}(\mathbf{n}_1) + \mathcal{O}(r_1) \ , \tag{4.3}$$

where ε_{ij} is a function of $\mathbf{n}_1 = \frac{\mathbf{x}-\mathbf{y}_1}{r_1}$, the direction of approach to the singularity [ε_{ij} is given by Eq. (A.3a)]. From Eq. (4.3), the linearized source-term $\Sigma_{(1)}$ when $r_1 \to 0$ is like some $\Sigma_{(1)} = \mathcal{O}\left(\frac{1}{r_1^3}\right)$. Now by integrating "term by term" the expansion of $\Sigma_{(1)}$, we get a similar expansion, but which starts at the order $\mathcal{O}\left(\frac{1}{r_1}\right)$. Clearly, the solution $\sigma_{(1)}$, when $r_1 \to 0$, will be composed of the latter expansion, which represents a particular solution, and augmented by a possible *homogeneous* solution, solving the (source-free) Laplace equation. However, because our original Poisson equation $\Delta\sigma_{(1)} = \Sigma_{(1)}$ is satisfied in the sense of distributions, the only possible homogeneous solution we can add is one of the type "regular at the origin", taking the form of a sum of STF products $\hat{x}_L \equiv \mathrm{STF}(x_{i_1} \cdots x_{i_\ell})$. Because it is regular when $r_1 \to 0$, this homogeneous solution does not modify the leading singular behaviour of $\sigma_{(1)}$, that we have therefore proved to be given by $\sigma_{(1)} = \mathcal{O}\left(\frac{1}{r_1}\right)$. And, again, this can be checked with Eqs. (3.11)-(3.12). The argument is easily generalized to any non-linear order n (by induction on n), thus we conclude that the non-linear perturbation σ diverges when $r_1 \to 0$, but not faster than $\sigma = \mathcal{O}(\frac{1}{r_1})$. As a result, the expansion of the conformal factor $\Psi = \psi + \sigma$ is of the type

$$\Psi = \frac{\zeta(\mathbf{n}_1)}{r_1} + \mathcal{O}\left(r_1^0\right) \ , \tag{4.4}$$

where ζ depends on the unit direction $\mathbf{n}_1$ but not on r_1. From this we deduce that $\gamma_{ij} = \Psi^4 \widetilde{\gamma}_{ij}$ behaves dominantly when $r_1 \to 0$ like

$$\gamma_{ij}(\mathbf{x}) = \left(\frac{\zeta(\mathbf{n}_1)}{r_1} \right)^4 \Big[\delta_{ij} + \varepsilon_{ij}(\mathbf{n}_1) + \mathcal{O}\left(r_1\right) \Big] \,. \tag{4.5}$$

See Eq. (A.4a) for the expression of $\zeta(\mathbf{n}_1)$ at linearized order.

Let us now perform some inversion of the radial coordinate r_1. We consider the coordinate change, valid in a neighbourhood of the particle 1, that is defined by $\mathbf{x} \to (\rho_1, \mathbf{n}_1)$, where

$$\rho_1 = \frac{\zeta^2(\mathbf{n}_1)}{r_1} \,. \tag{4.6}$$

First we have $dx^i = dr_1^i$, where $r_1^i \equiv (r_1, \mathbf{n}_1)$, because the particle is at rest: $v_1^i = 0$. Then, the coordinate transformation $r_1^i \to \rho_1^i \equiv (\rho_1, \mathbf{n}_1)$ involves simply the change of the radial variable given by Eq. (4.6), with the angular part $\mathbf{n}_1$ being unchanged[11]. We compute

$$dr_1^i \;=\; \left(\frac{\zeta}{\rho_1} \right)^2 \left(\delta^{ij} - 2 n_1^i n_1^j + 2 n_1^i \chi^j \right) d\rho_1^j \,, \tag{4.7a}$$

$$\chi^j \;\equiv\; \left(\delta^{jk} - n_1^j n_1^k \right) \frac{\partial \ln \zeta}{\partial n_1^k} \,. \tag{4.7b}$$

(Notice that $n_1^j \chi^j = 0$.) We then find that, in the new coordinate system $\rho_1^i = (\rho_1, \mathbf{n}_1)$, the metric, say Γ_{ij}, admits when $\rho_1 \to +\infty$ the *bounded* expansion announced in Eq. (4.2), in which the quantity $\pi_{ij}(\mathbf{n}_1)$ reads

$$\pi_{ij} \equiv -4 n_1^{(i} \chi^{j)} + 4 \chi^i \chi^j + (\delta^{ki} - 2 n_1^k n_1^i + 2 n_1^k \chi^i)(\delta^{lj} - 2 n_1^l n_1^j + 2 n_1^l \chi^j) \varepsilon_{kl} \,. \tag{4.8}$$

It is such that $n_1^i n_1^j \varepsilon_{ij} = n_1^i n_1^j \pi_{ij}$. [See also Eqs. (A.5b) and (A.6).] This completes the proof of Theorem 2.

[11] More general coordinate transformations, involving a change of the unit direction $\mathbf{n}_1$, are also possible.

B. The ADM mass

Because the metric is asymptotically flat at spatial infinity (Theorem 2), the binary's total ADM mass is given by the usual surface integral on a topological 2-sphere at infinity:

$$M = \frac{c^2}{16\pi G} \lim_{r\to+\infty} \int dS^i \Big(\partial_j \gamma_{ij} - \partial_i \gamma_{jj}\Big) \ , \tag{4.9}$$

where dS^i is the outward surface element on the surface at infinity ($dS^i = d\Omega r^2 n^i$ in the case of a coordinate sphere). To be more explicit about M, we recall from Section IV A that some functions A_{ij} and X of the unit direction $\mathbf{n} = \mathbf{x}/r$ exist so that, when $r \to +\infty$,

$$\beta s_{ij} = \frac{A_{ij}(\mathbf{n})}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \ , \tag{4.10a}$$

$$\sigma = \frac{X(\mathbf{n})}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \ . \tag{4.10b}$$

Here A_{ij} is simply linear in β, while X is in the form of a full non-linear series in β: $X = \beta X_{(1)} + \beta^2 X_{(2)} + \cdots$ (see also the Appendix). In terms of these functions M is given by

$$M = \frac{c^2}{G}\left\{2(\alpha_1+\alpha_2) + \int \frac{d\Omega}{4\pi}\left[2X + \frac{1}{2} n^i n^j A_{ij}\right]\right\} \ . \tag{4.11}$$

The first term is identical to the result in the Brill-Lindquist solution.

The computation of Eq. (4.11) is quite straightforward at the level of the *linearized* approximation, thanks to the closed-form expression (3.11). However, our aim will be the computation of the 2PN energy in Section V, and we can ascertain beforehand (by counting the required powers of G), that among all the terms contributing to the 2PN energy there must be one coming from the *second-order* perturbation, solution of Eq. (3.13). We find that this particular non-linear term is to be computed only in M (not in the individual masses m_1, m_2). It comes from that part of $\sigma_{(2)}$ — or, more precisely, of its leading order coefficient $X_{(2)}$ (when $r \to +\infty$) — which does not involve the constants α_1 and α_2, i.e. the part which would be the analogue of the first term in $\sigma_{(1)}$ as given by Eq. (3.11). The other parts, proportional to α_1 or α_2, appear at the 3PN order at least.

We succeeded in obtaining this non-linear term thanks to the method described by Eq. (3.14). First, one can check that the source-term behaves like $\Sigma_{(2)} = \mathcal{O}\left(\frac{1}{r^4}\right)$, so there is

no direct contribution, computed "term-by-term" from the expansion of the source-term, to the multipole expansion at the order $\mathcal{O}(1/r)$: i.e. the first term in the right side of (3.14) is at least $\mathcal{O}(1/r^2)$. The looked-for non-linear contribution is therefore given directly by the value of the "mass monopole" of $\sigma_{(2)}$, i.e. the integral appearing in the second term of Eq. (3.14) for $\ell = 0$. So,

$$\sigma_{(2)} = -\frac{1}{4\pi r}\int_{\mathbb{R}^3} d^3\mathbf{y}\,\Sigma_{(2)}(\mathbf{y}) + \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{4.12}$$

The integral (4.12) can be computed analytically [12], and yields the following contribution to the ADM mass:

$$2X_{(2)} = -\frac{1}{2\pi}\int_{\mathbb{R}^3} d^3\mathbf{y}\,\Sigma_{(2)}(\mathbf{y}) = \frac{1}{128 r_{12}^3} + \mathcal{O}(\alpha) , \tag{4.13}$$

where the remainder $\mathcal{O}(\alpha)$ indicates that the terms proportional to α_1 or α_2 (in this second-order perturbation $\propto \beta^2$), are not to be considered for the present calculation. We thereby obtain M with sufficient accuracy for controlling the energy at the 2PN order [13]. The result reads

$$M = \frac{c^2}{G}\left\{2(\alpha_1+\alpha_2) + \frac{\beta(\alpha_1+\alpha_2)}{24 r_{12}^2} + \frac{\beta^2}{r_{12}^3}\left[\frac{1}{128} + \mathcal{O}(\alpha)\right] + \mathcal{O}\left(\beta^3\right)\right\} . \tag{4.14}$$

The cubic and higher-order perturbations $\mathcal{O}(\beta^3)$ are neglected. See the Appendix for the explicit expansion coefficients needed in this computation.

C. The black-hole masses

The situation as concerns the two individual masses m_1 and m_2 is less easy than with the ADM mass because we dispose only of the notion of "asymptotic finiteness" in the vicinity of the particles, described by the fall-off property (4.2). Nevertheless, we wish to find an appropriate concept for the black-hole masses. What we shall do is to *define* m_1 by the same

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[12] This integral is convergent, thus it is unnecessary to include a finite-part operation *FP* (for notational simplicity we skip the multipole-expansion symbol $\mathcal{M}$).

[13] This means, by the way, that the relative accuracy on M itself is actually 3PN — because of the rest-mass contribution.

formula as for the ADM mass M, but using the coordinate system $\rho_1^i = (\rho_1, \mathbf{n}_1)$ in the limit where $\rho_1 \to +\infty$. Accordingly we pose

$$m_1 = \frac{c^2}{16\pi G} \lim_{\rho_1 \to +\infty} \int dS_1^i \left(\frac{\partial \Gamma_{ij}}{\partial \rho_1^j} - \frac{\partial \Gamma_{jj}}{\partial \rho_1^i} \right) \quad \text{and} \quad 1 \leftrightarrow 2 \;, \tag{4.15}$$

where $\Gamma_{ij}(\rho_1, \mathbf{n}_1)$ denotes the metric in the coordinates ρ_1^i, with ρ_1 being shown in Eq. (4.6), and where the outward surface element reads $dS_1^i = d\Omega_1 \rho_1^2 n_1^i$ in the case of a coordinate sphere.

Because Γ_{ij} is not asymptotically flat in the usual sense, due to the term with π_{ij} in Eq. (4.2), the mass defined by the previous integral will be typically unbounded (i.e. m_1 will in general tend toward infinity like ρ_1). Therefore, the definition (4.15) does not *a priori* make sense. A possibility would be to discard the divergent part ($\propto \rho_1$) of the mass, and thereby to consider only the finite part of the integral in Eq. (4.15), i.e. the coefficient of the zeroth power of ρ_1 in the expansion at infinity. This could represent an appropriate postulate for the mass in a general situation [the finite part prescription would be similar to the *FP* present in Eq. (3.14)]. However such a "finite part" process imposed into the definition of the mass appears to represent (untill more convincing justification is proposed) a somewhat artificial and *ad-hoc* recipe [14]. Gladly enough, we shall not need to invoke any *ad-hoc* finite part because we shall prove that if we restrict ourselves to the computation of the binary's energy at the 2PN order, which represents anyway the maximal order at which our solution is physically relevant, the masses m_1 and m_2 needed in this computation *are* perfectly well-defined.

To compute m_1 and m_2, we must control the expansion of the metric when $r_1 \to 0$ to one order beyond Eqs. (4.3)-(4.4). Let us pose

$$\beta s_{ij} = \varepsilon_{ij}(\mathbf{n}_1) + r_1 \mu_{ij}(\mathbf{n}_1) + \mathcal{O}(r_1^2) \;, \tag{4.16a}$$

$$\Psi = \frac{\zeta(\mathbf{n}_1)}{r_1} + \eta(\mathbf{n}_1) + \mathcal{O}(r_1) \;. \tag{4.16b}$$

[14] The ADM mass, given by the surface term that arises in the Hamiltonian, has been obtained even for space-times that are not asymptotically flat [29]. However, it seems difficult to apply this result in the present case, notably because of our lack of knowledge of the lapse function N in the vicinity of the particles (after radial inversion $r_1 \to \rho_1$ of the coordinates).

[The expressions of the coefficients are relagated to Eqs. (A.3)-(A.4) in the Appendix.] It is straightforward to derive from this the expression of the metric Γ_{ij} in the coordinate system $(\rho_1, \mathbf{n}_1)$, taking into account the term $\sim 1/\rho_1$ in Eq. (4.2):

$$\Gamma_{ij} = \left(1 + 4\frac{\zeta\eta}{\rho_1}\right)\left[\delta_{ij} + \pi_{ij}\right] + \frac{\zeta^2}{\rho_1}\kappa_{ij} + \mathcal{O}\left(\frac{1}{\rho_1^2}\right) \ , \tag{4.17}$$

where $\kappa_{ij} = (\delta^{ki} - 2n_1^k n_1^i + 2n_1^k \chi^i)(\delta^{lj} - 2n_1^l n_1^j + 2n_1^l \chi^j)\mu_{kl}$. Then, by inserting it into Eq. (4.15), we obtain

$$m_1 = \frac{c^2}{G}\lim_{\rho_1\to+\infty}\int \frac{d\Omega_1}{4\pi}\left[-\frac{1}{4}\rho_1\pi_{jj} + 2\eta\zeta\big(1 + n_1^i n_1^j \varepsilon_{ij}\big) + \frac{1}{2}\zeta^2 n_1^i n_1^j \mu_{ij}\right] \ . \tag{4.18}$$

Here, π_{jj} denotes the trace of the quantity (4.8), and we have used the facts that $n_1^i n_1^j \pi_{ij} = n_1^i n_1^j \varepsilon_{ij}$ and $n_1^i n_1^j \kappa_{ij} = n_1^i n_1^j \mu_{ij}$, together with the useful cancellation of the angular integral

$$\int \frac{d\Omega_1}{4\pi} n_1^i n_1^j \varepsilon_{ij}(\mathbf{n}_1) = 0 \ , \tag{4.19}$$

which can be checked from Eq. (A.4a) in the Appendix.

We observe that Eq. (4.18) contains a term proportional to ρ_1, with π_{jj} as a factor, which can and *does* make the mass to become infinite in the limit $\rho_1 \to +\infty$ [indeed, see Eq. (A.7)]. As suggested above, such an infinite term could be removed by some procedure of taking the finite part (whose meaning would *a priori* be quite unclear), but we find that this infinite term is in fact "negligible" for the present purpose, because π_{jj} starts only at non-linear order in β,

$$\pi_{jj} = 4\chi^j\chi^j + 4\big(n_1^k\chi^l + n_1^k n_1^l \chi^j\chi^j\big)\varepsilon_{kl} = \mathcal{O}\left(\beta^2\right) \ . \tag{4.20}$$

[See also the expression (A.6).] Therefore we get, at the linearized level, a finite expression for the mass (formally):

$$m_1 = \frac{c^2}{G}\int \frac{d\Omega_1}{4\pi}\left[2\eta\zeta\big(1 + n_1^i n_1^j \varepsilon_{ij}\big) + \frac{1}{2}\zeta^2 n_1^i n_1^j \mu_{ij}\right] + \mathcal{O}\left(\beta^2\right) \ . \tag{4.21}$$

This expression is sufficient to control the 2PN energy. The non-linear corrections $\mathcal{O}\left(\beta^2\right)$ are infinite but will not be needed in this paper. In a sense, they do not belong to the "realm" of the present solution, which is limited to the physics at the 2PN approximation: i.e. agreement with the 2PN metric in the near-zone, and internal consistency of the asymptotics

up to the 2PN order as regards the energy content of the solution (see Section V)[15]. As for the linearized terms in Eq. (4.21), they are dealt with thanks to the explicit result (3.11), and we get

$$m_1 = \frac{2c^2\alpha_1}{G}\left[1 + \frac{\alpha_2}{r_{12}} + \frac{\beta}{48r_{12}^2}\left(1 - \frac{25\alpha_2}{r_{12}}\right) + \mathcal{O}\left(\beta^2\right)\right] \quad \text{and} \quad 1 \leftrightarrow 2\,. \tag{4.22}$$

By setting $\beta = 0$ into Eq. (4.22), we reproduce the result valid in the case of the Brill-Lindquist solution.

V. THE BINARY'S GEOMETROSTATIC ENERGY

With the masses being defined and computed "geometrically" in Section IV, we get the opportunity of an interesting consistency check of our solution, concerning the "geometrostatic" energy that is associated with those masses. Indeed, in Theorem 1 we recovered the 2PN metric in the near zone, therefore we know the coordinate transformation $\delta x^i = \xi^i$ between our presently used coordinate system and the harmonic one. The gauge transformation vector has been obtained in Eq. (2.8). Therefore it is possible to control, without ambiguity — because we determined the coordinate transformation —, the binary's interacting energy E up to the same 2PN order. If our solution has anything cogent (physically speaking), the latter energy should be in complete agreement with the post-Newtonian result at the 2PN order known from Refs. [30, 31, 32, 33].

Theorem 3 *The binary's interacting energy, deduced from those masses (4.9) and (4.15) computed by surface integrals at infinity, i.e.*

$$\frac{E}{c^2} = M - m_1 - m_2\,, \tag{5.1}$$

gives back the energy calculated by standard post-Newtonian methods at the 2PN *order (after invoking the same coordinate transformation as in Theorem 1).*

[15] Recall that our proposal for binary black-hole initial data is to adopt the conformal metric (2.1) and to deduce the conformal factor Ψ from it by *numerical* methods. The present *analytic* investigation [including the theoretical definition of the mass (4.15)] is aimed at verifying the physical soundness of this proposal.

The ADM mass has been previously obtained in the form (4.14), while the masses m_1 and m_2 follow from the result (4.22). The corresponding energy E reads then

$$E = \frac{c^4}{G}\left\{-\frac{4\alpha_1\alpha_2}{r_{12}}\left(1-\frac{25\beta}{48r_{12}^2}\right)+\frac{\beta^2}{r_{12}^3}\left[\frac{1}{128}+\mathcal{O}\left(\alpha\right)\right]+\mathcal{O}\left(\beta^3\right)\right\} . \tag{5.2}$$

As we said above, this formula is accurate enough to get full control of the 2PN approximation. We also remarked that at this order the energy is well-defined (i.e. finite).

Our task is to expand that energy when $c \to +\infty$, using the facts that α_1, $\alpha_2 = \mathcal{O}\left(\frac{1}{c^2}\right)$ and $\beta = \mathcal{O}\left(\frac{1}{c^4}\right)$ [actually, we already applied the post-Newtonian approximation when arguing that the remainder $\mathcal{O}\left(\alpha\right)$ is negligible]. Also, we know that β is given by Eq. (3.1a). We first invert Eq. (4.22), in an iterative post-Newtonian way, so as to obtain α_1, α_2 with relative 2PN accuracy. The result is

$$\alpha_1 = \frac{Gm_1}{2c^2}\left[1-\frac{Gm_2}{2r_{12}c^2}+\frac{G^2m_2(5m_1+3m_2)}{12r_{12}^2c^4}+\mathcal{O}\left(\frac{1}{c^6}\right)\right] \quad \text{and} \quad 1 \leftrightarrow 2\, . \tag{5.3}$$

At the 1PN order, we are consistent with the Brill-Lindquist prediction and with what is given by Eq. (3.4). But, quite naturally, we find that Eq. (5.3) differs from the Brill-Lindquist result at the 2PN order [16].

Substituting (5.3) back into (5.2), we then arrive, after suitable post-Newtonian expansion, at the expression of the 2PN energy in terms of the two "physical" individual masses:

$$E = -\frac{Gm_1m_2}{r_{12}}+\frac{G^2m_1m_2(m_1+m_2)}{2r_{12}^2c^2}-\frac{G^3m_1m_2(m_1^2+19m_1m_2+m_2^2)}{4r_{12}^3c^4}+\mathcal{O}\left(\frac{1}{c^6}\right) . \tag{5.4}$$

It is composed of the standard Newtonian potential energy (for static black holes), augmented by 1PN and 2PN corrections. The result, however, is not yet the one we want to prove, because the particles' separation r_{12} corresponds to our particular coordinate system. If we want to compare E with the post-Newtonian prediction, we must take into account the coordinate transformation (in the "near zone") that was determined in Theorem 1.

[16] In the case of the Brill-Lindquist solution we have

$$\alpha_1^{\mathrm{BL}} = \frac{Gm_1}{2c^2}\left[1-\frac{Gm_2}{2r_{12}c^2}+\frac{G^2m_2(m_1+m_2)}{4r_{12}^2c^4}+\mathcal{O}\left(\frac{1}{c^6}\right)\right] .$$

The required link between r_{12} and the particles' separation R_{12} in harmonic coordinates is readily obtained with the help of Eq. (2.8), which permits to compute the shift of the world-lines that is induced by the coordinate transformation [17]. We get

$$r_{12} = R_{12} - \frac{G^2(m_1^2+m_2^2)}{4R_{12}c^4} + \mathcal{O}\left(\frac{1}{c^6}\right) \,. \tag{5.5}$$

The result we find after replacing Eq. (5.5) into Eq. (5.4), and effecting the post-Newtonian re-expansion [the replacement is to be made only into the Newtonian term of (5.4)], is

$$E = -\frac{Gm_1m_2}{R_{12}} + \frac{G^2m_1m_2(m_1+m_2)}{2R_{12}^2c^2} - \frac{G^3m_1m_2(2m_1^2+19m_1m_2+2m_2^2)}{4R_{12}^3c^4} + \mathcal{O}\left(\frac{1}{c^6}\right) \,. \tag{5.6}$$

Most satisfactorily, we discover that (5.6) is in complete agreement with the prediction for the 2PN energy in harmonic coordinates, calculated in Refs. [30, 31, 32, 33]. The expression we end up with is the same as given by Eq. (B6) in Ref. [32] — in which of course one must set the two particles' velocities to zero.

The latter agreement is interesting, but we said that it is quite mandatory if our solution makes sense. Technically speaking, it necessitated the control of the metric up to second-order perturbation theory on the Brill-Lindquist background [*cf.* the crucial contribution of Eq. (4.13) to the ADM mass]. Furthermore, the result (5.6) can be said to check the global character of the solution (notably the asymptotics therein) — because the physical masses have been computed by surface integrals at infinity. Alternatively, it shows the relevance at 2PN order of the definition (4.15) we adopted for the black-hole individual masses.

Finally, let us comment that Theorem 3 tells us something about the physical tenets at the basis of the usual post-Newtonian approximation when it is applied to the description of black holes. Indeed, the "post-Newtonian" masses m_1 and m_2, which parametrize the post-Newtonian iteration, are introduced as being the coefficients of Dirac delta-functions in the Newtonian density of point-like particles [14]. Now, we have just seen that in fact

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[17] We discard an infinite "self-term" when considering the coordinate-transformation vector $\xi_i(\mathbf{x})$ at the singular location of the two particles. Thus, from Eq. (2.8), which is valid *a priori* only outside the singularities, we obtain the shift vector

$$\xi_i(\mathbf{y}_1) = \frac{G^2m_2^2}{4c^4}\frac{n_{12}^i}{r_{12}} + \mathcal{O}\left(\frac{1}{c^6}\right) \,.$$

In other words we consider only the finite part of $\xi_i(\mathbf{x})$ when $\mathbf{x} \to \mathbf{y}_1$ (in Hadamard's sense).

these post-Newtonian masses agree (within the present approach at least, i.e. in the time-symmetric situation, and up to the 2PN order) with the “geometrostatic” masses which are associated with some Einstein-Rosen-like bridges. We view this as a confirmation that the post-Newtonian calculations, which treat formally the compact objects by means of delta-function singularities [18], are appropriate to the description of systems of black holes (as long as the *orbital* motion of the black holes can be considered to be “slow” in the post-Newtonian sense, i.e. in the so-called inspiralling phase of black-hole binaries).

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APPENDIX: RELEVANT EXPANSION FORMULAS

In this Appendix we provide the explicit expressions of the various coefficients in the expansion formulas introduced in Section IV for the computation of the masses.

For the ADM mass we need the dominant term, when $r \to +\infty$, in the generating function s_{ij} and the solution of the linear-order perturbation for the conformal factor [see Eqs. (4.10)]. They are given by

$$A_{ij} = \frac{\beta}{2r_{12}} n_{12}^{<i} n_{12}^{j>} \,, \tag{A.1a}$$

$$\beta X_{(1)} = \frac{\beta}{r_{12}} \left\{ -\frac{1}{32} \left[(\mathbf{n}.\mathbf{n}_{12})^2 - \frac{1}{3} \right] + \frac{1}{48} \frac{\alpha_1 + \alpha_2}{r_{12}} \right\} \,, \tag{A.1b}$$

where $\mathbf{n} = \mathbf{x}/r$ and $\mathbf{n}_{12} = (\mathbf{y}_1 - \mathbf{y}_2)/r_{12}$ (with $\mathbf{n}.\mathbf{n}_{12}$ denoting the ordinary scalar product). The second-order perturbation coefficient was already computed in Eq. (4.13):

$$\beta^2 X_{(2)} = \frac{\beta^2}{r_{12}^3} \left[\frac{1}{256} + \mathcal{O}(\alpha) \right] \,. \tag{A.2}$$

[18] For instance, the calculation of the 3PN equations of motion of compact binaries reported in Ref. [33].

For the masses m_1 and m_2 we require the expansions when $r_1 \to 0$ of s_{ij} and the conformal factor to one order beyond the dominant term [see Eqs. (4.16)]. For s_{ij} we have

$$\varepsilon_{ij} = \frac{\beta}{r_{12}^2}\left(n_{12}^{<i}n_{12}^{j>} - \frac{1}{2}n_1^{<i}n_{12}^{j>}\right) , \tag{A.3a}$$

$$\mu_{ij} = \frac{\beta}{r_{12}^3}\left[n_{12}^{<i}n_{12}^{j>}\left(-\mathbf{n}_1.\mathbf{n}_{12} - \frac{3}{4}\right) + n_1^{<i}n_{12}^{j>}\left(\frac{3}{4}\mathbf{n}_1.\mathbf{n}_{12} + \frac{3}{4}\right) - \frac{1}{4}n_1^{<i}n_1^{j>}\right] . \tag{A.3b}$$

For the conformal factor:

$$\zeta = \alpha_1\left[1 + \frac{\beta}{r_{12}^2}\left(-\frac{1}{2}(\mathbf{n}_1.\mathbf{n}_{12})^2 + \frac{5}{24}\mathbf{n}_1.\mathbf{n}_{12} + \frac{1}{6}\right)\right] + \mathcal{O}\left(\beta^2\right) , \tag{A.4a}$$

$$\begin{aligned}\eta = 1 + \frac{\alpha_2}{r_{12}} + \frac{\beta}{r_{12}^2}\Bigg[&-\frac{1}{24}\mathbf{n}_1.\mathbf{n}_{12} + \frac{1}{48}\\ &+\frac{\alpha_1}{r_{12}}\left(\frac{1}{4}(\mathbf{n}_1.\mathbf{n}_{12})^3 + \frac{5}{32}(\mathbf{n}_1.\mathbf{n}_{12})^2 - \frac{5}{24}\mathbf{n}_1.\mathbf{n}_{12} - \frac{5}{96}\right)\\ &+\frac{\alpha_2}{r_{12}}\left(-\frac{1}{24}\mathbf{n}_1.\mathbf{n}_{12} - \frac{25}{48}\right)\Bigg] + \mathcal{O}\left(\beta^2\right) .\end{aligned} \tag{A.4b}$$

The quantities χ^i and π_{ij} defined by Eqs. (4.7b) and (4.8) read (at linear order in β)

$$\chi^i = \frac{\beta}{r_{12}^2}\left(\mathbf{n}_1.\mathbf{n}_{12} - \frac{5}{24}\right)\left[-n_{12}^i + (\mathbf{n}_1.\mathbf{n}_{12})n_1^i\right] + \mathcal{O}\left(\beta^2\right) , \tag{A.5a}$$

$$\pi_{ij} = \frac{\beta}{r_{12}^2}\left(n_{12}^{<i}n_{12}^{j>} - \frac{1}{3}n_1^{<i}n_{12}^{j>} - \frac{1}{6}(\mathbf{n}_1.\mathbf{n}_{12})n_1^{<i}n_1^{j>}\right) + \mathcal{O}\left(\beta^2\right) . \tag{A.5b}$$

Note that at linear order π_{ij} is trace-free. Its trace arises only at second-order in β,

$$\pi_{jj} = \frac{\beta^2}{r_{12}^4}\left(-\frac{1}{6}\mathbf{n}_1.\mathbf{n}_{12} + \frac{5}{144}\right)\left[(\mathbf{n}_1.\mathbf{n}_{12})^2 - 1\right] + \mathcal{O}\left(\beta^3\right) . \tag{A.6}$$

At this order the angular average of π_{ij} is non-zero:

$$\int \frac{d\Omega_1}{4\pi}\pi_{jj} = -\frac{5}{216}\frac{\beta^2}{r_{12}^4} + \mathcal{O}\left(\beta^3\right) . \tag{A.7}$$

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